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HW 6

Math 575B
April 24, 2008

1 Theory.

1. The elliptic equation

$$\nabla^2\phi + d\phi_x + e\phi_y + f\phi = 0 \tag{1}$$

in x and y is satisfied by the function $\phi(x,y)$ in the area Ω. Prove that a non-constant value of ϕ cannot assume a positive maximum or a negative minimum inside of Ω when f is negative.

Proof. Assume ϕ is non-constant on Ω. Let $\nabla^2\phi|_{(x_0,y_0)} = a$ where (x_0,y_0) is an extremum. $\phi_x|_{(x_0,y_0)} = \phi_y|_{(x_0,y_0)} = 0$

Let (x_0,y_0) be a maximum, $a < 0$.

$$a = -f\phi \tag{2}$$

but, f is negative which implies $\phi(x_0,y_0)$ must be negative.

Now, let (x_0,y_0) be a minimum, $a > 0$.

$$a = -f\phi \tag{3}$$

but, f is negative which implies $\phi(x_0,y_0)$ must be positive.

Therefore, ϕ cannot assume a positive maximum or a negative minimum inside of Ω when f is negative. □

2. Show that by integrating the Poisson equation about the origin on a disk of radius $\epsilon = \Delta r << 1$, the value of $u_{0,j} \equiv u_0$ can be approximated by

$$u_0 = \frac{1}{J}\sum_{j=1}^{J} u_{1,j} - f(0)(\frac{\Delta r}{2})^2 \tag{4}$$

Moreover, show that this is a second-order accurate approximation.

Solution. Begin by integrating Laplace's equation over a small circle of radius ϵ.

$$\int\int_{D_\epsilon} \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} r dr d\theta = \int\int_{D_\epsilon} f r dr d\theta \tag{5}$$

$$\int_0^{2\pi} \epsilon\frac{\partial u}{\partial r} d\theta = f(0)\frac{\epsilon^2}{2}\int_0^{2\pi} d\theta \tag{6}$$

$$\int_0^{2\pi} \epsilon\frac{\partial u}{\partial r} d\theta = f(0)\epsilon^2\pi \tag{7}$$

where circle radius is sufficiently small to justify evaluation of f at $r = 0$. The second integral with the $\frac{\partial^2 u}{\partial \theta^2}$ term goes to 0 since ϵ is chosen sufficiently small so that the θ dependence of u goes away. Use approximation to integral on the LHS and substitute $\epsilon = \frac{\Delta r}{2}$ and $\partial r = \Delta r$ $\partial\theta = \Delta\theta$.

$$f(0)(\frac{\Delta r}{2})^2\pi = \sum_{j=1}^{J} \frac{u_{1,j} - u_0}{\Delta r}\frac{\Delta r}{2}\Delta\theta \tag{8}$$

$$= \sum_{j=1}^{J}(u_{1,j} - u_0)\frac{1}{2}\Delta\theta \tag{9}$$

Make substitution $\Delta\theta = \frac{2\pi}{J}$.

$$f(0)(\frac{\Delta r}{2})^2\pi = \sum_{j=1}^{J}(u_{1,j} - u_0)\frac{\pi}{J} \tag{10}$$

$$f(0)(\frac{\Delta r}{2})^2 = \frac{1}{J}(\sum_{j=1}^{J} u_{1,j} - Ju_0) \tag{11}$$

Solve for u_0:

$$u_0 = \frac{1}{J}\sum_{j=1}^{J} u_{1,j} - f(0)(\frac{\Delta r}{2})^2 \tag{12}$$

This is a second-order approximation due to discretization of the integral and derivative.

2 Experiment.

1. Do the "worked out example of FEM in the notes"

   Solution. Picked grid points:

x	y
0	2
$\sqrt{2}$	$\sqrt{2}$
2	0
1	0
$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
0	1
0.75	1.299
1.299	0.75

Next, made node number assignments (counterclockwise), found local stiffness matrices, combined into a global stiffness matrix, and applied Dirichlet boundary conditions to come up with the following stiffness matrix for points 2, 7 and 8 from above table.

$$A = \begin{pmatrix} 2.6137 & -1.685 & -1.685 \\ -1.685 & 4.5049 & -0.2516 \\ -1.685 & -0.2516 & 4.5049 \end{pmatrix}$$

Call above matrix A. Solve problem $Av = F$. Where $F \equiv$ force term:

$$F = \begin{pmatrix} 0.5708 \\ 0 \\ 0 \end{pmatrix}$$

and v is the vector of coefficients in the expansion of u using piecelinear functions. Construct three piecelinear basis functions for the solution of the matrix problem:

$$\phi_{1_i} = \frac{1}{2\det B}[(x_{1_{i+1}}y_{1_{i+2}} - x_{1_{i+2}}y_{1_{i+1}}) - (y_{1_{i+1}} - y_{1_{i+2}})x + (x_{1_{i+2}} - x_{1_{i+1}})y] \quad (13)$$

$$\phi_{2_i} = \frac{1}{2\det B}[(x_{2_{i+1}}y_{2_{i+2}} - x_{2_{i+2}}y_{2_{i+1}}) - (y_{2_{i+1}} - y_{2_{i+2}})x + (x_{2_{i+2}} - x_{2_{i+1}})y] \quad (14)$$

$$\phi_{3_i} = \frac{1}{2\det B}[(x_{3_{i+1}}y_{3_{i+2}} - x_{3_{i+2}}y_{3_{i+1}}) - (y_{3_{i+1}} - y_{3_{i+2}})x + (x_{3_{i+2}} - x_{3_{i+1}})y] \quad (15)$$

where

$$B = \begin{pmatrix} 1 & 1 & 1 \\ \sqrt{2} & 0.75 & 1.299 \\ \sqrt{2} & 1.299 & 0.75 \end{pmatrix}$$

Solve for v:

$$v = \begin{pmatrix} 0.4464 \\ 0.1768 \\ 0.1768 \end{pmatrix}$$

The solution compares rather poorly to the exact solution $\frac{4}{17}(r^2 - \frac{1}{r^2})\sin 2\theta$. The difference between the approximate and actual solutions at the grid points is:

$v_{approximate}$	v_{actual}
0.0603	0
0.0058	0.8824
0.0603	0
0.1261	0
0.0988	0
0.1261	0
0.0571	0.3679
0.0571	0.3679

2. Solve Poisson's equation on a unit disk with periodic boundary conditions and boundary value:

$$u(1, \theta) = \phi(\theta) = \frac{1}{2} + \sin(2\theta) \quad (16)$$

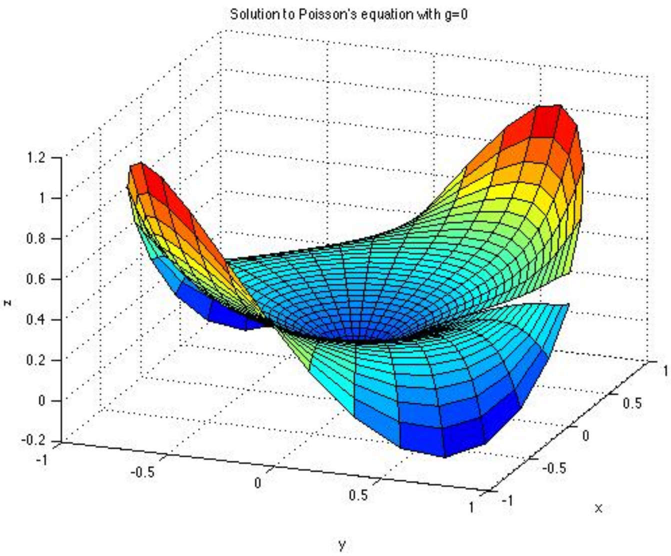

Figure 1: Solution of Laplace's equation with above boundary conditions

Solution. The plot of the solution for $d = 30$, radial discretization, and $m = 32$ To show convergence of the method, the mesh in d is successively cut in half. The resulting plot is:

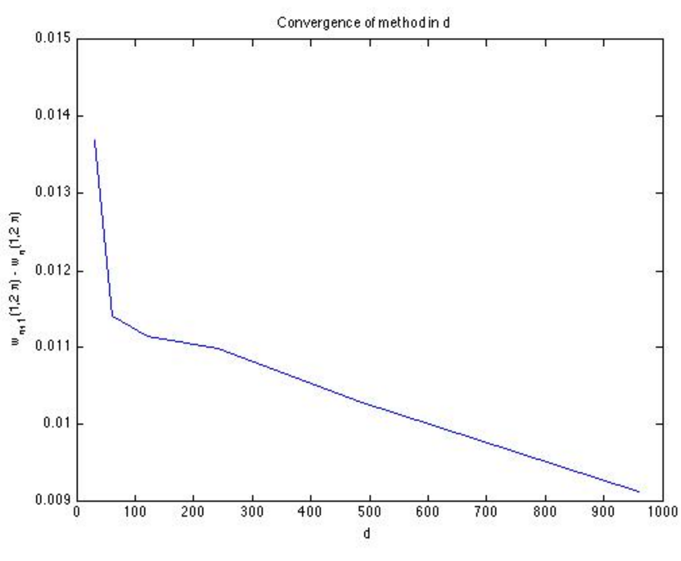

Figure 2: Convergence of method in d

So the method converges as d becomes larger and larger. The code for the solution is included on the next page.